Splitting: Tanaka's SDE revisited

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What follows is my attempt to understand a set of ideas being developed by Boris Tsirelson. I do this by studying a specific, and I hope interesting, example.

Tanaka's SDE is one of the easiest examples of a stochastic differential equation with no strong solution. Suppose $\big(X_t; t \geq 0\big)$ is a real-valued Brownian motion starting from zero and we put $B_t = \int_0^t sgn(X_s)dX_s$ then B is also a Brownian motion and Tanaka's SDE

$$X_t = \int_0^t sgn(X_s)dB_s, \tag{1}$$

is satisfied. But the trajectory of B does not determine that of X. Recall Tanaka's formula

$$|X_t| = \int_0^t sgn(X_s)dX_s + L_t, \tag{2}$$

where $\big(L_t; t \geq 0\big)$ is the local time process of X at zero. We find

$$|X_t| = B_t + \sup_{s \leq t}(-B_s) \tag{3}$$

but B does not tell us the signs of the excursions from zero made by X.

In a discrete time framework things, work out differently. Let $\big(X_n; n \geq 0\big)$ be the symmetric nearest neighbour random walk on the integers. Define $sgn(a)$ to be $+1$ if $a \geq 0$ and -1 if $a < 0$, and then let $Z_0 = 0$ and $Z_{n+1} - Z_n = sgn(X_n)\big(X_{n+1} - X_n\big)$ then $\big(Z_n; n \geq 0\big)$ is again a symmetric random walk and we may write a discrete version of equation (1):

$$X_n = \sum_{k=0}^{n-1} sgn(X_k)\big(Z_{k+1} - Z_k\big). \tag{4}$$

The equations (2) and (3) have discrete time versions:

$$|X_n + \tfrac{1}{2}| - \tfrac{1}{2} = \sum_{k=0}^{n-1} sgn(X_k)\big(X_{k+1} - X_k\big) + L_n \tag{5}$$

where $L_0 = 0$ and for $n \geq 1$ we define $L_n = \sum_{k=0}^{n-1} 1_{\big(X_k, X_{k+1} \in \{0,-1\}\big)}$, and,

$$|X_n + \tfrac{1}{2}| - \tfrac{1}{2} = Z_n + \sup_{k \leq n}\big(-Z_k\big). \tag{6}$$

The halves appear because of the lack of symmetry in our definition of sgn- it is not something to worry about. This time Z does determine X: the information about whether X_n is below or above $-\frac{1}{2}$- which at first sight appears to be missing

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in (6)- is coded in the following way. Find the last time $r \in \{0, 1, \ldots, n\}$ such that $Z_r = -\sup_{k \le r}\big(-Z_k\big)$. If this r is even then $X_n + \frac{1}{2}$ is positive while if it is odd then $X_n + \frac{1}{2}$ is negative.

This note is concerned with understanding what happens to this precious information about the sign of X when we try to obtain the continuous-time model by taking scaling limits of the discrete model. This is inspired by work of Boris Tsirelson on the spectra of noises and stability- see [4], [5], and [2].

One way to understand why the information about the signs does not survive the passage to the limit is to observe that it is noise sensitive. Instead of one copy of the random walk take a pair (Z, Z') that are ρ-correlated where $\rho \in (0, 1)$. This means that together they form a nearest-neigbour random walk on the lattice $\mathbb{Z}^2$ with $\mathbb{E}\big[(Z_{k+1} - Z_k)(Z'_{k+1} - Z'_k)\big] = \rho$. We think of Z' as being a perturbation of Z. Now construct X and X' from Z and Z' so that equation (4) and its prime version hold. As n becomes large (as it will when we try to take scaling limits) we find that $sgn(X_n)$ and $sgn(X'_n)$ become uncorrelated no matter how strong the (fixed) correlation ρ between Z and Z' is. Thus, in a certain sense, $sgn(X_n)$ is asymptotically sensitive to small perturbations of Z.

The discussion of the previous paragraph, although very elementary, is (a variant of) the observation that eventually led Tsirelson to profound results in the theory of filtrations [6]. There good account of the story in [2].

Next instead of constructing Z' by perturbing the whole path of Z we may only perturb some sections. More precisely let $A \subset [0, 1]$ be a finite union of closed intervals with (to be safe) dyadic rational end-points. Fix $\rho \in (0, 1)$. For each n construct a random walk $\big((Z_k, Z'_k); 0 \le k \le n\big)$ on $\mathbb{Z}^2$ with

$$\mathbb{E}\big[(Z_{k+1} - Z_k)(Z'_{k+1} - Z'_k)\big] = \begin{cases} \rho & k2^{-n} \in A \\ 1 & k2^{-n} \in A^c. \end{cases}$$

Now as before construct X and X' and consider the correlation of $sgn(X_n)$ and $sgn(X'_n)$. This time as n tends towards infinity we obtain a nontrivial limit which we denote by $\phi(\rho, A)$.

We write the Wiener chaos expansion of any random variable belonging to $\mathcal{L}^2(B)$ in the form

$$\hat{f}_0 + \int_0^1 \hat{f}_1(s) dB_s + \int_0^1 \int_0^{s_2} \hat{f}_2(s_1, s_2) dB_{s_1} dB_{s_2} + \ldots$$

Then we construct a finite measure on $\cup_{n \ge 0}\big\{(s_1, s_2, \ldots, s_n) \in [0, 1]^n | s_1 < s_2 \ldots < s_n\big\}$ having density $|\hat{f}_n(s_1, \ldots, s_n)|^2$ with respect to Lebesgue measure. We call this the spectral measure of the random variable whose chaos expansion we used. If we start with variable having $\mathcal{L}^2$-norm equal to one this measure is a probability measure- and we can think of it as determining the law of a finite random subset S of $[0, 1]$. Thus there is a probability $|\hat{f}_0|^2$ that S is empty, a probability $|\hat{f}_1(s)|^2 ds$ that it contains a single point lying in $(s, s + ds)$ and so on. Suppose for a moment that the information on signs did survive in the limit, and that the X satisfying Tanaka's SDE was some functional of the Brownian motion B. Then we could apply this construction to $sgn(X_1)$- and obtain a random subset S. If A is a fixed subset of $[0, 1]$ once more

then let $|S \cap A|$ denote the number of points of S lying in A. Then it is reasonable to expect that $\phi(\rho, A)$ would be given by

$$\phi(\rho, A) = \mathbb{E}\left[\rho^{|S\cap A|}\right]. \tag{7}$$

The $\mathbb{E}$ appearing here is with respect to the law of S which does not live on the same probability space as X and B.

The discussion of the preceding paragraph is based on a false premise, but nevertheless there is a random subset- still denoted by S- such that equation (7) holds. This subset possess, with probability one, an infinite number of elements. If we take $A = [0, 1]$ then $|S \cap A|$ is infinity and $\rho^\infty = 0$ (by definition if you like!). Thus $\phi(\rho, [0, 1])$ is 0 for any ρ- this is just the sensitivity to noise property with which we began. In what follows we examine the law of this S more closely. Not surprisingly the Wiener chaos expansion is our principle tool.

For any $x > 0$ and $t \in (0, 1)$ let $m_{(t,x)}$ denote the spectral measure of

$$1\left(\sup_{h\in[t,1]}(B_t - B_h) < x\right).$$

Note that the total mass of this measure is just $\mathbb{P}\left(\sup_{u\in[t,1]}(B_t - B_u) < x\right) < 1$, but we will nevertheless speak of a random set S having distribution $m_{(t,x)}$. This subset is supported on $[t, 1]$. Let $q_h(x, dy)$ denote the (defective) transition probability distributions of Brownian motion killed on hitting 0.

Lemma 1. *Suppose that $0 < s < t < 1$ and that S is distributed according to $m_{(s,x)}$. Then the subset $S \cap [t, 1]$ is distributed according to*

$$\int q_{t-s}(x, dy) m_{(t,y)}.$$

Now suppose that $\left(\lambda_t(dx); 0 < t \leq 1\right)$ is an entrance law for killed Brownian motion; thus, for any $0 < s < t \leq 1$,

$$\lambda_t(dy) = \int q_{t-s}(x, dy)\lambda_s(dx).$$

We may define a family of measures $m_{(t,\lambda)}$ for $t \in (0, 1)$ via

$$m_{(t,\lambda)} = \int \lambda_t(dy) m_{(t,y)},$$

and by virtue of the lemma they have a certain consistency property- that is - if S is distributed according to $m_{(s,\lambda)}$ then for any $t > s$ the intersection $S\cap[t, 1]$ is distributed according to $m_{(t,\lambda)}$. Note that the total mass of $m_{(t,\lambda)}$ is the 'probability' that the killed BM survives to time 1 when it is started according to λ- this does not depend on t. Because of this consistency property there is a random set $S \subset (0, 1]$ whose distribution we denote by m_λ whose intersection with $[t, 1]$ has distribution $m_{(t,\lambda)}$ for any t. This S may be infinite- there is the possibility of 0 being an accumulation point.

From this point on we will take $(\lambda_t; 0 < t \leq 1)$ to be a multiple of the entrance law for the Itô excursion measure for the positive excursions of Brownian motion. Choose this multiple so that m_λ becomes a probability measure. More explicitly we have

$$\lambda_t(dy) = y\, t^{-3/2} \exp\left\{-\frac{y^2}{2t}\right\} dy \qquad y > 0. \tag{8}$$

Recall that a random variable is said to be arc-sine distributed if it has distribution

$$s(dt) = \frac{dt}{\pi\sqrt{t(1-t)}} 1_{[0,1]}(t)\, dt.$$

The time at which a BM attains its minimum between times 0 an 1 is so distributed.

Theorem 2. *The limits $\phi(\rho, A)$ exist and admit the following description. Take two random subsets S_1 and S_2 distributed according to m_λ and a $[0,1]$-valued random variable g with the arc-sine distribution . Suppose that S_1, S_2 and g are independent. Take*

$$S = g(1 - S_1) \cup ((1-g)S_2 + g),$$

then for all ρ and A

$$\phi(\rho, A) = \mathbb{E}\left[\rho^{|S \cap A|}\right].$$

Proof of Lemma. Begin by writing

$$1\left(\sup_{h \in [s,1]} (B_s - B_h) < x\right) = 1\left(\sup_{h \in [s,t]} (B_s - B_h) < x\right) \times 1\left(\sup_{h \in [t,1]} (B_t - B_h) < x + B_t - B_s\right).$$

Condition on $(B_r; r \leq t)$ and then replace the second factor with its Wiener chaos expansion and so obtain an expansion of which the typical term is

$$\int_t^1 dB_{h_1} \int_t^{h_1} dB_{h_2} \ldots \int_t^{h_{k-1}} dB_{h_k} \, 1\left(\sup_{h \in [s,t]} (B_s - B_h) < x\right) \hat{f}_k(t, x + B_t - B_s | h_1, \ldots, h_k).$$

We now replace each integrand by its chaos expansion- this must simply result in the chaos expansion of

$$1\left(\sup_{h \in [s,1]} (B_s - B_h) < x\right).$$

On comparing the two expansions it may be seen that if S is distributed according to $m_{(s,x)}$ then $S \cap [t,1]$ contains exactly k points at positions $(h_1, h_1 + dh_1)$ through $(h_k, h_k + dh_k)$ with probability

$$\mathbb{E}\left[1\left(\sup_{h \in [s,t]} (B_s - B_h) < x\right) |\hat{f}_k(t, x + B_t - B_s | h_1, \ldots, h_k)|^2\right] dh_1 \ldots dh_k,$$

but since $|\hat{f}_k(t,y|h_1,\dots,h_k)|^2 dh_1 \dots dh_k$ is just the probability distribution of S under $m_{(t,y)}$ we are done. □

Proof of Theorem. Stage 1. Fix an admissible subset A. For each n consider the correlated random walk $\big((Z_k, Z'_k); 0 \le k \le n\big)$. There is the usual weak convergence in the space of continuous $\mathbb{R}^2$-valued paths to a process $\big((B_t, B'_t); 0 \le t \le 1\big)$, each component of which forms a one-dimensional Brownian motion and their co-variation is simply:

$$dB_t dB'_t = \begin{cases} \rho dt & t \in A \\ dt & t \in A^c. \end{cases}$$

Let g be the time at which B attains its minimum between times 0 and 1, and similarly define g'. Now the correlation of $sgn(X_n)$ and $sgn(X'_n)$ can be split into the sum of two contributions. One arises when the random walks Z and Z' attain their minimum (between times 0 and n) values simultaneously - in this case $sgn(X_n)$ and $sgn(X'_n)$ are equal. The remaining contribution tends to zero for large n and so the limits $\phi(\rho, A)$ exist and are given by

$$\phi(\rho, A) = \mathbb{E}\left[1_{(g=g')}\right].$$

Stage 2. The two random times g and g' can only be equal if their common value lies in one of the components of A^c. For each such component we consider the common time at which B and B' attain their respective minimum (over that component) and compute the probability that this is actually the global minimum of both Brownian motions. We obtain

$$\mathbb{E}\left[1_{(g=g')}\right] = \int_{A^c}\int_0^\infty\int_0^\infty p(u_t, v_t; dt, dy_1, dy_2) m_{(v_t,y_2)}\left[\rho^{|S\cap A|}\right] m_{(1-u_t,y_1)}\left[\rho^{|(1-S)\cap A|}\right],$$

where

$$u_t = \sup\{h < t : h \in A\},$$
$$v_t = \inf\{h > t : h \in A\},$$

and $p(u, v; dt, dy_1, dy_2)$ is the law of the triple

$$\big(g(u,v), B_u - I(u,v), B_v - I(u,v)\big),$$

$g(u,v)$ denoting the time at which B attains its minimum $I(u,v) = \inf\{h \in [u,v] : B_h\}$.

Stage 3. By virtue of the scaling properties of BM we have

$$m_{(v,y)}\left[\rho^{|S\cap A|}\right] = m_{\left((v-t)/(1-t),\, y/\sqrt{1-t}\right)}\left[\rho^{|(t+(1-t)S)\cap A|}\right]$$
$$m_{(1-u,y)}\left[\rho^{|(1-S)\cap A|}\right] = m_{\left((t-u)/t,\, y/\sqrt{t}\right)}\left[\rho^{|t(1-S)\cap A|}\right].$$

A well-known exercise (Revuz and Yor [1], chapter XII) confirms that

$$\begin{aligned}p(u,v;dt,dy_1,dy_2) &= \frac{dt}{\pi}\lambda_{t-u}(dy_1)\lambda_{v-t}(dy_2)\\ &= s(dt)\lambda_{(t-u)/t}(dy_1/\sqrt{t})\lambda_{(v-t)/(1-t)}(dy_2/\sqrt{1-t}).\end{aligned}$$

Putting these into the formula obtained in the previous section and recalling the definition of m_λ we obtain the desired result:

$$\mathbb{E}\left[1_{(g=g')}\right] = \int_{A^c} s(dt)m_\lambda\left[\rho^{|(t+(1-t)S)\cap A|}\right]m_\lambda\left[\rho^{|t(1-S)\cap A|}\right].$$

$\square$

It is possible to generalise the family of measures $m_{(t,x)}$ from which we obtained m_λ. Starting from a bounded function f defined on $\mathbb{R}_+$ we may expand

$$f\big(B_1 - B_t + x\big)1\left(\sup_{h\in[t,1]}(B_t - B_h) < x\right),$$

and whence construct a measure m_λ^f. I would like to know when such measures corresponding to different f are equivalent and in this case how to compute the Radon-Nikodým density. This is part of the problem of obtaining the spectral resolution (see [3]) of the noise of splitting. This is a noise richer than white noise: in addition to the increments of a Brownian motion B it carries a countable collection of independent Bernoulli random variables which are attached to the local minima of B.

References

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